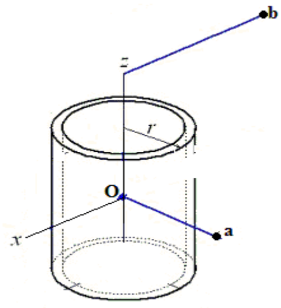
POTENCIAL ELÉCTRICO, ENERGIA E CAPACIDADE

ELECTRIC POTENTIAL, ENERGY AND CAPACITY

1) Considere uma linha recta segundo z carregada com uma distribuição linear de cargas de 5nC/m. Consider a straight line oriented by z loaded with a linear electrical charge distribution of 5nC/m, see picture at left.



a) Determinar a diferença de potencial entre os pontos **a** e **b** tais que: a(2m,π/2,0) e b(4m,π,5m). Find the electrical potential difference between the points **a** and **b** at (2m,π/2,0) and at (4m,π,5m), respectively.

$$\mathrm{V_A - V_B} = -\int_{\mathrm{B}}^{\mathrm{A}} \vec{\mathrm{E}} \cdot d\vec{\mathrm{L}} \qquad \vec{\mathrm{E}} = \frac{\rho_l}{2\pi\varepsilon \mathrm{R}}\vec{\mathrm{a}}_R \qquad d\vec{l} = dr\vec{a}_r + rd\phi\vec{a}_\phi + dz\vec{a}_z$$

Resolução Resolution

$$\mathrm{V_A - V_B} = -\int_{(4,\pi,5)}^{(2,\pi/2,0)} \frac{5\times10^{-9}}{2\pi 8.854\times10^{-12} r}\vec{\mathrm{a}}_r \cdot (dr\vec{a}_r + rd\phi\vec{a}_\phi + dz\vec{a}_z)$$

$$\mathrm{V_{AB}} = -\int_{(4,\pi,5)}^{(2,\pi/2,0)} \frac{5\times10^{-9}}{2\pi 8.854\times10^{-12} r}(dr + 0 + 0) \qquad \mathrm{V_{AB}} = -\frac{5\times10^{-9}}{2\pi 8.854\times10^{-12}}\int_4^2 \frac{1}{r}dr$$

$$\mathrm{V_{AB}} = -\frac{5\times10^{-9}}{2\pi 8.854\times10^{-12}}\left[\ln(r)\right]_4^2 \qquad \mathrm{V_{AB}} = -\frac{5\times10^{-9}}{2\pi 8.854\times10^{-12}}(\ln(2) - \ln(4))$$

$$\mathrm{V_{AB}} = -\frac{5\times10^{-9}}{2\pi 8.854\times10^{-12}}\ln\left(\frac{1}{2}\right) \qquad \mathrm{V_{AB}} = \frac{5\times10^{-9}}{2\pi 8.854\times10^{-12}}\ln(2)$$

Resposta Answer

$$\mathrm{V_{AB}} \cong 62.3\,\mathrm{V}$$

b) Determinar a diferença de potencial entre os pontos a e b tais que: a(2m,0, 0) e b(4m, 0, 0). Find the electrical potential difference between the points a and b at (2n, 0, 0) and at (4m, 0, 0), respectively.

Resolução Resolution

$$V_A - V_B = -\int_{(4,0,0)}^{(2,0,0)} \frac{5\times10^{-9}}{2\pi 8.854\times10^{-12} r}\vec{a}_r \cdot (dr\vec{a}_r + rd\phi\vec{a}_\phi + dz\vec{a}_z)$$

$$V_{AB} = -\int_{(4,0,0)}^{(2,0,0)} \frac{5\times10^{-9}}{2\pi 8.854\times10^{-12} r}(dr + 0 + 0) \qquad V_{AB} = -\frac{5\times10^{-9}}{2\pi 8.854\times10^{-12}}\int_4^2 \frac{1}{r}dr$$

$$V_{AB} = -\frac{5\times10^{-9}}{2\pi 8.854\times10^{-12}}[\ln(r)]_4^2 \qquad V_{AB} = -\frac{5\times10^{-9}}{2\pi 8.854\times10^{-12}}(\ln(2) - \ln(4))$$

$$V_{AB} = -\frac{5\times10^{-9}}{2\pi 8.854\times10^{-12}}\ln\left(\frac{1}{2}\right) \qquad V_{AB} = \frac{5\times10^{-9}}{2\pi 8.854\times10^{-12}}\ln(2)$$

Resposta Answer

$$V_{AB} \cong 62.3\,\mathrm{V}$$

c) Compare e comente os resultados obtidos nas duas alíneas anteriores. Compare and discuss the results obtained in the two previous questions.

Resolução Resolution

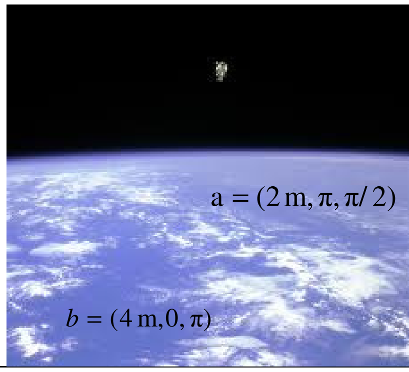
Neste exemplo a diferença de potencial apenas depende da componente segundo a radial dos pontos em causa. In this example the difference of potential depends only on the radial component of the points.

Resposta Answer

O resultado é o mesmo. The result is the same.

2) Numa zona do espaço o campo eléctrico é definido pela equação ao lado em coordenadas esféricas. In an area of the space the electrical field is defined by the equation at left in a spherical frame.

$$\vec{E} = -\frac{16}{r^2}\vec{a}_r \quad \text{V/m}$$



a) Determine a diferença de potencial entre os pontos **a** e **b**. Find the difference of electrical potential between the points **a** and **b**.

$$V_A - V_B = -\int_B^A \vec{E}\cdot d\vec{L} \qquad \vec{E} = -\frac{16}{r^2}\vec{a}_r \quad \text{V/m} \qquad d\vec{l} = dr\vec{a}_r + rd\theta\vec{a}_\theta + r\sin(\theta)\, d\phi\vec{a}_\phi$$

Resolução Resolution

$$V_A - V_B = -\int_B^A \vec{E}\cdot d\vec{L} \qquad V_{AB} = -\int_{(4,0,\pi)}^{(2,\pi,\pi/2)} \left(-\frac{16}{r^2}\vec{a}_r\right)\cdot\left(dr\vec{a}_r + rd\theta\vec{a}_\theta + r\sin(\theta)\, d\phi\vec{a}_\phi\right)$$

$$V_{AB} = -\int_{(4,0,\pi)}^{(2,\pi,\pi/2)} \left(-\frac{16}{r^2}\right)(dr + 0 + 0) \qquad V_{AB} = \int_4^2 \frac{16}{r^2} dr \qquad V_{AB} = -16\left[\frac{1}{r}\right]_4^2 \qquad V_{AB} = -4\,\text{V}$$

Resposta Answer

$$V_{AB} = -4\,\text{V}$$

b) Mostre que, no presente caso, a diferença de potencial apenas é função de uma coordenada e comente este facto. Show that, in this case, the difference of potential is a function of only one coordinate and discuss this.

Resolução Resolution

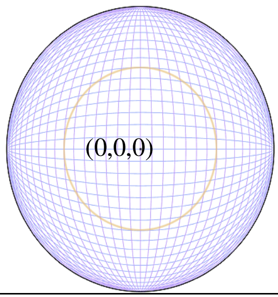
$$V_A - V_B = -\int_B^A \vec{E}\cdot d\vec{L} \qquad V_{AB} = -\int_{(R_B,\theta_B,\varphi_B)}^{(R_A,\theta_A,\varphi_A)} \left(-\frac{16}{r^2}\vec{a}_r\right)\cdot\left(dr\vec{a}_r + rd\theta\vec{a}_\theta + r\sin(\theta)\, d\phi\vec{a}_\phi\right)$$

$$V_{AB} = -\int_{(R_B,\theta_B,\varphi_B)}^{(R_A,\theta_A,\varphi_A)} \left(-\frac{16}{r^2}\right)(dr + 0 + 0) \qquad V_{AB} = \int_{R_B}^{R_A} \frac{16}{r^2} dr \qquad V_{AB} = -16\left[\frac{1}{r}\right]_{R_B}^{R_A}$$

$$V_{AB} = -16\left(\frac{1}{R_A} - \frac{1}{R_B}\right)$$

Resposta Answer

V_{AB} apenas é função de R_A e de R_B, por isso, apenas depende uma coordenada que é a coordenada radial. V_{AB} is only a function of R_A and R_B therefore only depends on one coordinate that is the radial coordinate.

3) Uma superfície esférica de raio 2m com centro em (0,0,0) tem um potencial eléctrico dado pela equação apresentada ao lado. A spherical surface with radius 2m and center in (0,0,0) has an electrical potential given by the equation shown at left.	$V = V_0; \quad r < a$ $V = a\frac{V_0}{r}; \quad r \geq a$ 

a) Calcule a diferença de potencial entre os pontos P1 e P2 que distam da origem 5m e 3m, respectivamente. Find the electrical potential difference between two points P1 and P2 that are distant from the origin of 5m and 3m, respectively.

$$V = V_0; \quad r < a$$
$$V = a\frac{V_0}{r}; \quad r \geq a$$

a = 2m

Resolução Resolution

$$V_1 = 2\frac{V_0}{5} \qquad V_2 = 2\frac{V_0}{3} \qquad V_{12} = V_1 - V_2 \qquad V_{12} = 2\frac{V_0}{5} - 2\frac{V_0}{3}$$

$$V_{12} = -\frac{4}{15}V_0$$

Resposta Answer

$$V_{12} = -\frac{4}{15}V_0$$

b) Determine a expressão do campo eléctrico **E** representado pela equação de potencial dada. Find the expression of the electric field **E** represented by the given equations of electrical potential.

$$V = V_0; \quad r < a$$
$$V = a\frac{V_0}{r}; \quad r \geq a$$

$$\vec{E} = -\nabla V$$

$$grad(A) = \nabla A = \frac{\partial A}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial A}{\partial \theta}\vec{a}_\theta + \frac{\partial A}{r\,\text{sen}\theta\,\partial \phi}\vec{a}_\phi$$

Resolução Resolution

$$\vec{E} = \begin{cases} -\nabla V_0 & r < a \\ -\nabla a\frac{V_0}{r} & r \geq a \end{cases} \qquad \vec{E} = \begin{cases} 0 & r < a \\ -aV_0\frac{\partial}{\partial r}\frac{1}{r}\vec{a}_r & r \geq a \end{cases} \qquad \vec{E} = \begin{cases} 0 & r < a \\ a\frac{V_0}{r^2}\vec{a}_r & r \geq a \end{cases}$$

Resposta Answer

$$\vec{E} = \begin{cases} 0 & r < a \\ a\frac{V_0}{r^2}\vec{a}_r & r \geq a \end{cases}$$

c) Calcule a totalidade da energia armazenada no campo eléctrico. Find the total electrical energy stored in the electric field.

$$W_E = \frac{1}{2}\int_v \varepsilon E^2 dv \qquad \vec{E} = \begin{cases} 0 & r < a \\ a\frac{V_0}{r^2}\vec{a}_r & r \geq a \end{cases} \qquad dv = r^2\sin(\theta)\,dr\,d\theta\,d\phi$$

Resolução Resolution

$$W_E = \varepsilon\frac{(aV_0)^2}{2}\lim_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_a^R \frac{1}{r^4}r^2\sin(\theta)dr d\theta d\phi \qquad W_E = \varepsilon\frac{(aV_0)^2}{2}\lim_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\left[-r^{-1}\right]_a^R\sin(\theta)d\theta d\phi$$

$$W_E = -\varepsilon\frac{(aV_0)^2}{2}\lim_{R\to\infty}\left(\frac{1}{R}-\frac{1}{a}\right)\int_0^{2\pi}\int_0^{\pi}\sin(\theta)d\theta d\phi \qquad W_E = \varepsilon\frac{(aV_0)^2}{2}\frac{1}{a}\int_0^{2\pi}\int_0^{\pi}\sin(\theta)d\theta d\phi$$

$$W_E = \varepsilon\frac{(aV_0)^2}{2}\frac{1}{a}\int_0^{2\pi}\left[-\cos(\theta)\right]_0^{\pi}d\phi \qquad W_E = \varepsilon\frac{aV_0^2}{2}\int_0^{2\pi}2d\phi$$

$$W_E = 2\pi\varepsilon a V_0^2$$

Resposta Answer

$$W_E = 2\pi\varepsilon a V_0^2$$

d) Utilizando o operador divergente, identifique a densidade de carga nas diferentes zonas. Using the divergent operator identify the charge density in different areas.

$$\nabla\cdot\vec{D} = \rho \qquad \vec{D} = \varepsilon\vec{E} \qquad \vec{E} = \begin{cases} 0 & r < a \\ a\dfrac{V_0}{r^2}\vec{a}_r & r \geq a \end{cases}$$

$$div(\vec{A}) = \nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\,\mathrm{sen}\,\theta}\frac{\partial}{\partial\theta}\left(A_\theta \mathrm{sen}\,\theta\right) + \frac{1}{r\,\mathrm{sen}\,\theta}\frac{\partial A_\phi}{\partial\phi}$$

Resolução Resolution

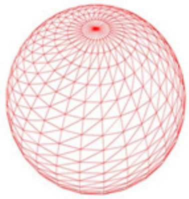
$$\nabla\cdot\vec{E} = \frac{1}{r^2}\frac{\partial\left(r^2 E_r\right)}{\partial r} + \frac{1}{r\,\mathrm{sen}\,\theta}\frac{\partial}{\partial\theta}\left(E_\theta \mathrm{sen}\,\theta\right) + \frac{1}{r\,\mathrm{sen}\,\theta}\frac{\partial E_\phi}{\partial\phi}$$

$$\rho = \nabla\cdot\varepsilon\vec{E} = \begin{cases} \nabla\cdot\varepsilon 0 & r < a \\ \nabla\cdot\left(\varepsilon a\dfrac{V_0}{r^2}\vec{a}_r\right) & r > a \end{cases} \qquad \rho = \begin{cases} 0 & r < a \\ \varepsilon\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 a\dfrac{V_0}{r^2}\right) & r > a \end{cases}$$

$$\rho = \begin{cases} 0 & r < a \\ \varepsilon a V_0\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(1) & r > a \end{cases} \qquad \rho = \begin{cases} 0 & r < a \\ 0 & r > a \end{cases}$$

Resposta Answer

$$\rho = \begin{cases} 0 & r < a \\ 0 & r > a \end{cases}$$

4) Deduza a expressão da capacidade de uma casca esférica isolada de raio r que se encontra inserida num meio de permissividade relativa Er. Derive the expression of the electric capacity of an isolated spherical shell with radius r that is inserted through a relative permittivity Er.	 $d\vec{E} = \frac{\rho \vec{a}_r}{4\pi\varepsilon R^2} ds$

$$V_A - V_B = -\int_B^A \vec{E} \cdot d\vec{L} \qquad \vec{D} = \varepsilon\vec{E} \qquad C = \left|\frac{Q}{V}\right| \qquad \rho = \frac{dQ}{ds} \qquad d\vec{E} = \frac{\rho \vec{a}_r}{4\pi\varepsilon r^2} ds$$

$$ds_{\phi\theta} = r^2 \sin(\theta) d\theta d\phi \qquad d\vec{l} = dr\vec{a}_r + rd\theta\vec{a}_\theta + r\sin(\theta)\, d\phi\vec{a}_\phi$$

Resolução Resolution

Procurar a carga armazenada na casca esférica. Find the electrical charge stored in a spherical surface.

$$\rho = \frac{dQ}{ds} \qquad dQ = \rho ds \qquad dQ = \rho r^2 \sin(\theta) d\theta d\phi \qquad Q = \rho r^2 \int_0^{2\pi} \int_0^{\pi} \sin(\theta) d\theta d\phi$$

$$Q = \rho r^2 \int_0^{2\pi} 2 d\phi \qquad Q = 4\pi r^2 \rho$$

Procurar o potencial da esfera em relação ao infinito. Find the potential of the sphere relative to infinity.

No infinito um corpo finito pode ser visto como pontual, então o campo eléctrico pode ser modelizado pela expressão da carga pontual. At the infinity a finite body can be seen as timely. Then the electric field can be modeled by the expression of the point charge.

$$\vec{E} = \frac{Q}{4\pi\varepsilon R^2}\vec{a}_R \qquad \vec{E} = \frac{r^2\rho}{\varepsilon R^2}\vec{a}_R \qquad V_{AB} = -\lim_{R\to\infty}\int_r^R \left(\frac{r^2\rho}{\varepsilon R^2}\vec{a}_R\right)\cdot(dR\vec{a}_R + rd\theta\vec{a}_\theta + r\sin(\theta)\, d\phi\vec{a}_\phi)$$

$$V_{AB} = \rho\frac{r^2}{\varepsilon}\lim_{R\to\infty}\left[\frac{1}{R}\right]_r^R \qquad V_{AB} = \rho\frac{r^2}{\varepsilon}\lim_{R\to\infty}\left(\frac{1}{R} - \frac{1}{r}\right) \qquad V_{AB} = -\rho\frac{r}{\varepsilon}$$

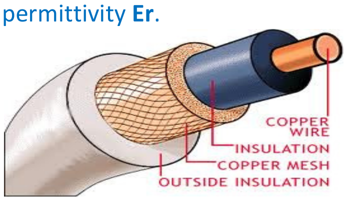
$$C = \left|\frac{4\pi r^2 \rho}{-\rho\frac{r}{\varepsilon}}\right| \qquad C = 4\pi\varepsilon r \qquad C = 4\pi\varepsilon_r\varepsilon_0 r$$

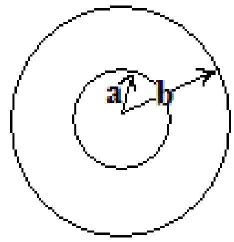
O mesmo seria conseguido tendo em conta que um corpo carregado simetricamente pode ser visto como uma carga pontual de valor igual sua carga. The same would be achieved taking into account that a body symmetrically charged can be seen as a point charged with the same value.

Resposta Answer

$$C = 4\pi\varepsilon_r\varepsilon_0 r$$

5) Deduza a expressão da capacidade por metro do cabo coaxial com raio interno, **a**, e raio externo, **b**, respectivamente. O dialéctico isolante entre o condutor interno e o externo tem permissividade relativa **Er**. Find the expression of the capacity per meter of the coaxial cable with inner and outer radius **a** and **b**, respectively. The insulating dialectic between the inner conductor and the outer conductor has relative permittivity **Er**.





$$C = \left|\frac{Q}{V}\right| \qquad V_A - V_B = -\int_B^A \vec{E}\cdot d\vec{L} \qquad \vec{D} = \varepsilon\vec{E} \qquad d\vec{l} = dr\vec{a}_r + rd\phi\vec{a}_\phi + dz\vec{a}_z$$

Resolução Resolution

Devido à simetria do condutor cilíndrico interno o campo eléctrico por ele gerado é o mesmo que o do condutor rectilíneo. Due to the symmetry of the internal cylindrical conductor the electric field generated by it is the same as the straight conductor.

Procurar a carga armazenada na linha. Find the electrical charge stored in the line.

$$\rho = \frac{dQ}{dl} \qquad dQ = \rho\, dl \qquad Q = \rho\int_0^L dl \qquad Q = \rho L$$

Procurar o potencial do condutor exterior em relação ao interior.

$$\vec{E} = \frac{\rho}{2\pi\varepsilon r}\vec{a}_R$$

$$V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{L} \qquad V_{ba} = -\int_a^b \left(\frac{\rho}{2\pi\varepsilon r}\vec{a}_R\right)\cdot(dr\vec{a}_r + rd\phi\vec{a}_\phi + dz\vec{a}_z) \qquad V_{ba} = -\int_a^b \frac{\rho}{2\pi\varepsilon r}dr$$

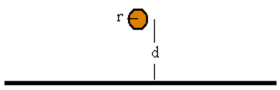
$$V_{ba} = -\frac{\rho}{2\pi\varepsilon}(\ln(b) - \ln(a)) \qquad V_{ba} = -\frac{\rho}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right) \qquad V_{ba} = \frac{\rho}{2\pi\varepsilon}\ln\left(\frac{a}{b}\right)$$

$$C = \left|\frac{Q}{V}\right| \qquad C = \left|\frac{\rho L}{\frac{\rho}{2\pi\varepsilon}\ln\left(\frac{a}{b}\right)}\right| \qquad C = \left|\frac{2\pi\varepsilon L}{\ln\left(\frac{a}{b}\right)}\right| \qquad C/L = \frac{2\pi\varepsilon}{\left|\ln\left(\frac{a}{b}\right)\right|} \qquad C/L = \frac{2\pi\varepsilon_R\varepsilon_0}{\left|\ln\left(\frac{a}{b}\right)\right|}$$

Resposta Answer

$$C/L = \frac{2\pi\varepsilon_R\varepsilon_0}{\left|\ln\left(\frac{a}{b}\right)\right|}$$

6) Encontre a expressão da capacidade por metro entre um condutor rectilíneo de raio r e um plano de massa que distam entre si de d. Para o cálculo, considere o diâmetro do condutor muito menor que a distância entre o condutor e o plano de massa. O diaIéctico isolante entre o condutor e o plano de massa tem permissividade relativa Er. Find the expression of the capacity per meter of a straight conductor of radius r and a ground plane that are distant from each other d. For the calculation, consider the conductor diameter much smaller than the distance between the conductor and the ground plane. The dialectical insulator between the conductor and the ground plane has relative permittivity Er.



$$\vec{E} = \frac{\rho}{2\pi\varepsilon R}\vec{a}_R \qquad C = \left|\frac{Q}{V}\right| \qquad V_A - V_B = -\int_B^A \vec{E}\cdot d\vec{L} \qquad \vec{D} = \varepsilon\vec{E} \qquad d\vec{l} = dr\vec{a}_r + rd\phi\vec{a}_\phi + dz\vec{a}_z$$

Resolução Resolution

Resposta Answer